

Seiberg-Witten Equations on Tubes

Liviu I. Nicolaescu*

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Introduction

In [10] we began a study of the 3-dimensional Seiberg-Witten equations on Seifert manifolds with two goals in mind: to compute the Seiberg-Witten-Floer (SWF) homology of these manifolds and ultimately, to produce gluing formulæ for the 4-dimensional Seiberg-Witten invariants.

A first difficulty one must overcome has to do with the less than obvious nature of the solutions of the 3-dimensional Seiberg-Witten equations. We dealt with this issue in [10] where we studied the behavior of these solutions as the Seifert fibration collapses onto its base (i.e. the background metric is shrunk in the fiber direction). As the metric is deformed, the solutions of the SW equations converge to solutions of some *adiabatic Seiberg-Witten equations*. These are variational equations and can be solved *explicitly*. Moreover, these adiabatic equations are very simple *zeroth order perturbations* of the original ones which suggests that the Morse theory for the adiabatic equations produces the same results as the original ones (which may have to be perturbed anyway to be placed in a generic framework).

A key fact established in [10] was that, in the case of a smooth S^1 -bundle N over a Riemann surface Σ equipped with a product-like metric with sufficiently short fibers, the adiabatic Morse function is Bott nondegenerate along the irreducible part of its critical set. This fact makes the adiabatic theory even more tempting to use for Floer theory computations.

The Bott extension of Morse theory (in the form described for example in [1]) describes a spectral sequence associated to a Morse-Bott function converging to the cohomology of the background manifold. This approach can be extended to our infinite dimensional situation as in [4] for the instanton homology.

The (co)boundary operators of this spectral sequence are defined in terms of the tunnelings between different components of the critical set, i.e. connecting trajectories of the gradient flow. In the case at hand, the gradient flow equations are

***Current address:** Dept. of Math., University of Michigan, Ann Arbor, MI 48109-1109, USA; liviu@math.lsa.umich.edu

in fact (zeroth order) perturbations of the 4-dimensional Seiberg-Witten equations on the tube $\mathbb{R} \times N$.

The main results of this paper (Theorem 2.1) will show that these tunnelings *do not exist* provided the fibers of N are sufficiently short. This implies the boundary operators are trivial and thus, the adiabatic theory leads to a *perfect* Morse function. To actually compute the (co)homology (as a *graded* object) one has to compute several spectral flows. We will address this issue elsewhere.

The proof of Theorem 2.1 is conceptually very simple. First of all, the results of [10] show that the critical points of the adiabatic Morse functional do not change as we shrink the fibers of N . In particular, there exists a *positive lower bound* for the L^∞ norms of these solutions and this bound is *independent of the shrinking geometry*. We next show that (when they exist) the tunnelings can be used to produce an *effective upper bound* for the L^∞ norms of the above critical points. This upper bound *converges to zero* as the fibers of N become shorter and shorter. Thus tunnelings cannot exist if N has short fibers. The key ingredient in the proof of this effective L^∞ -estimate is M\"oser's iteration technique in which we carefully keep track of the dependence of the best Sobolev embedding constants on the shrinking geometry.

Note After this paper was completed we learned of the paper [9] where these tunnelings are studied via algebraic geometric techniques. It is however not clear whether those results imply the adiabatic disappearance of tunnelings proved in this paper.

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1 Seiberg-Witten equations on smooth circle bundles

In this section we briefly survey some basic facts established in [10] (see also [9]) about the 3-dimensional Seiberg-Witten equations on smooth circle fibrations and then we introduce the 4-dimensional Seiberg-Witten equations on tubes.

§1.1 The differential geometric background Consider $\ell \in \mathbb{Z}$ and denote by $N = N_\ell$ the total space of a degree ℓ principal S^1 bundle over a compact oriented surface of genus g : $S^1 \hookrightarrow N_\ell \xrightarrow{\pi} \Sigma$. Denote by $\zeta \in \text{Vect}(N)$ the infinitesimal generator of the S^1 action. N has a natural orientation which can be described using any splitting $TN = \langle \zeta \rangle \oplus \pi^*T\Sigma$ determined by an arbitrary connection.

Assume Σ is equipped with a Riemann metric h_b such that $\text{vol}_{h_b}(\Sigma) = \pi$. Pick a connection form $\mathbf{i}\eta \in \mathfrak{i}\Omega^1(N)$ such that

$$-d\eta = 2\ell dv_{h_b}.$$

This choice is possible since $\frac{-1}{2\pi}d\eta$ represents the first Chern class of N which is ℓ . For each $\delta \geq 1$ define a metric h_δ on N by

$$h_\delta = \delta^{-2}\eta \otimes \eta \oplus \pi^*h_b.$$

When $\delta = 1$ we will write h instead of h_1 .

Using this metric we can orthogonally split $T^*N \cong \langle \eta \rangle \oplus \pi^*T^*\Sigma$ and this defines in a natural way an orientation on N . If $*_\delta$ denotes the Hodge $*$ operator of the metric h_δ we get

$$d\eta_\delta = 2\lambda_\delta *_\delta \eta_\delta$$

where $\eta_\delta = \delta^{-1}\eta$ and $\lambda_\delta = -\ell\delta^{-1}$. Again we set $\lambda = \lambda_1$.

Fix a local, oriented h -orthonormal coframe $\eta_0 = \eta, \eta_1, \eta_2$ on N and denote by $\zeta^0 = \zeta, \zeta^1, \zeta^2$ its dual frame. The bundle $\langle \eta \rangle^\perp$ has a natural complex structure locally defined by the correspondences $\eta_1 \mapsto -\eta_2 \mapsto -\eta_1$. In this way we get a complex line bundle $\mathcal{K} \rightarrow N$. It is isomorphic with the pullback of the canonical line bundle K_Σ of the base. We have a splitting

$$T^*N \otimes \mathbb{C} \cong \underline{\mathbb{C}} \oplus \mathcal{K} \oplus \mathcal{K}^{-1} \quad (1.1)$$

where $\underline{\mathbb{C}}$ denotes the trivial complex line bundle (over N). Note that the complex hermitian line bundle \mathcal{K} has *two* natural connections on it. A connection $\nabla = \nabla^h$ induced by the Levi-Civita connection of the metric h via the decomposition (1.1) and a connection ∇^∞ induced by pullback from the Levi-Civita connection on K_Σ .

For any complex vector bundle $E \rightarrow N$ we get

$$T^*N \otimes E \cong E \oplus \mathcal{K} \otimes E \oplus \mathcal{K}^{-1} \otimes E. \quad (1.2)$$

Thus, any connection $\nabla : C^\infty(E) \rightarrow C^\infty(T^*N \otimes E)$ naturally splits in three parts

- $\nabla_\zeta : C^\infty(E) \rightarrow C^\infty(E)$.
- ${}^b\nabla = \varepsilon \otimes (\nabla_1 - \mathbf{i}\nabla_2) : C^\infty(E) \rightarrow C^\infty(\mathcal{K} \otimes E)$
- ${}^b\bar{\nabla} = \bar{\varepsilon} \otimes (\nabla_1 + \mathbf{i}\nabla_2) : C^\infty(E) \rightarrow C^\infty(\mathcal{K}^{-1} \otimes E)$.

where $\varepsilon = 2^{-1/2}(\eta_1 + \mathbf{i}\eta_2)$, $\bar{\varepsilon} = 2^{-1/2}(\eta_1 - \mathbf{i}\eta_2)$ and $\nabla_j = \nabla_{\zeta^j}$ for $j = 0, 1, 2$. (Note that $\nabla = \eta \otimes \nabla_0 + 2^{-1/2}({}^b\nabla + {}^b\bar{\nabla})$).

Since TN is trivial, N admits *spin* structures and they are parameterized by the square roots of \mathcal{K} . We pick a square root of \mathcal{K} as a pullback of a fixed square root $K^{1/2}$ of K_Σ . Once this *spin* structure is fixed, the *spin*^c structures are parameterized bijectively by the family of isomorphism classes of complex line bundles on N .

The bundle of complex spinors \mathbb{S}_σ corresponding to the *spin*^c structure σ determined by a line bundle $\mathcal{L} = \mathcal{L}_\sigma \rightarrow N$ is

$$\mathbb{S} = \mathbb{S}_\sigma = \mathcal{K}^{-1/2} \otimes \mathcal{L} \oplus \mathcal{K}^{1/2} \otimes \mathcal{L}. \quad (1.3)$$

If A is a connection on \mathcal{L} then we can tensor it either with ∇^h , or with ∇^∞ to obtain two pairs of connections $(\nabla^A, \nabla^{A,\infty})$, a pair on $\mathcal{K}^{-1/2} \otimes \mathcal{L}$ and the other on $\mathcal{K}^{1/2} \otimes \mathcal{L}$. We can now construct the operators

$$\nabla_\zeta^A, \nabla_\zeta^{A,\infty} : C^\infty(\mathcal{K}^{\pm 1/2} \otimes \mathcal{L}) \rightarrow C^\infty(\mathcal{K}^{\pm 1/2} \otimes \mathcal{L})$$

$${}^b\nabla^A = {}^b\nabla^{A,\infty} : C^\infty(\mathcal{K}^{-1/2} \otimes \mathcal{L}) \rightarrow C^\infty(\mathcal{K}^{1/2} \otimes \mathcal{L})$$

and

$${}^b\overline{\nabla}^A = {}^b\overline{\nabla}^{A,\infty} : C^\infty(\mathcal{K}^{1/2} \otimes \mathcal{L}) \rightarrow C^\infty(\mathcal{K}^{-1/2} \otimes \mathcal{L})$$

Now form the operators $Z_A, Z_{A,\infty}, T_A : C^\infty(\mathbb{S}) \rightarrow C^\infty(\mathbb{S})$ which in terms of the splitting (1.3) have the block decompositions

$$Z_A = \begin{bmatrix} \mathbf{i}\nabla_\zeta^A & 0 \\ 0 & -\mathbf{i}\nabla_\zeta^A \end{bmatrix} \quad Z_{A,\infty} = \begin{bmatrix} \mathbf{i}\nabla_\zeta^{A,\infty} & 0 \\ 0 & -\mathbf{i}\nabla_\zeta^{A,\infty} \end{bmatrix}$$

$$T_A = \begin{bmatrix} 0 & {}^b\overline{\nabla}^A \\ {}^b\nabla^A & 0 \end{bmatrix}$$

These operators are formally selfadjoint and satisfy the following fundamental identities.

$$Z_{A,\infty} = Z_A + \lambda/2 = Z_A - \ell/2. \quad (1.4)$$

$$\{Z_A, T_A\} := Z_A T_A + T_A Z_A = -\lambda T_A + \mathbf{i} \begin{bmatrix} 0 & F_A^{0,1} \\ F_A^{1,0} & 0 \end{bmatrix} \quad (1.5)$$

where F_A denotes the curvature of A , $F_A^{0,1} = \bar{\varepsilon} \otimes F_A(\zeta, \zeta_1 + \mathbf{i}\zeta_2)$ and $F_A^{1,0} = \varepsilon \otimes F_A(\zeta, \zeta_1 - \mathbf{i}\zeta_2)$. The last two equalities imply

$$\{Z_{A,\infty}, T_A\} = \mathbf{i} \begin{bmatrix} 0 & F_A^{0,1} \\ F_A^{1,0} & 0 \end{bmatrix} \quad (1.6)$$

The Dirac operator on \mathbb{S} defined by the connection A is

$$\mathcal{D}_A = Z_A + T_A + \lambda = Z_A + T_A - \ell \quad (1.7)$$

Define the *adiabatic Dirac operator* by

$$\mathbf{D}_A = Z_{A,\infty} + T_A = \mathcal{D}_A - \lambda/2. \quad (1.8)$$

The above constructions depend on the parameter δ . For the metric h_δ one defines $Z_{A,\infty,\delta} = \delta Z_{A,\infty}$, $Z_{A,\delta} = Z_{A,\infty,\delta} - \lambda\delta/2$, $\mathcal{D}_{A,\delta} = Z_{A,\delta} + T_A + \lambda\delta$ and $\mathbf{D}_{A,\delta} = Z_{A,\infty,\delta} + T_A$.

§1.2 The 3-dimensional Seiberg-Witten equations The data entering the Seiberg-Witten equations are the following.

- (a) A spin^c structure σ on N determined by the line bundle $\mathcal{L} = \mathcal{L}_\sigma$.
- (b) A connection A of $\mathcal{L} \rightarrow N$.
- (c) A spinor ψ i.e. a section of the complex spinor bundle \mathbb{S}_σ associated to the given spin^c structure σ .

The connection A defines a geometric Dirac operator \mathcal{D}_A on \mathbb{S}_σ . The Seiberg-Witten equations are

$$(SW) : \begin{cases} \mathcal{D}_A \psi &= 0 \\ \mathbf{c}(*F_A) &= \tau(\phi) \end{cases}$$

where $*$ is the Hodge $*$ -operator of the metric g , $\tau(\psi) = \bar{\psi} \otimes \psi - \frac{1}{2}|\psi|^2 \in \text{End}(\mathbb{S})$ and \mathbf{c} is the Clifford multiplication defined in terms of the background metric h . If the metric h is replaced by the metric h_δ we get new equations $(SW)_\delta$ obtainable from (SW) via the substitutions

$$\mathcal{D}_A \mapsto \mathcal{D}_{A,\delta}, \quad * \mapsto *_\delta, \quad \mathbf{c} \mapsto \mathbf{c}_\delta.$$

The *adiabatic Seiberg-Witten equations* are

$$(SW)_\infty : \begin{cases} \mathbf{D}_A \psi &= 0 \\ \mathbf{c}(*F_A) &= \tau(\phi) \end{cases}$$

For each $\delta \geq 1$ we get an adiabatic equation $(SW)_{\infty,\delta}$ derived from $(SW)_\infty$ in the same way $(SW)_\delta$ is obtained from (SW) .

Using the decomposition $\mathbb{S} = \mathcal{K}^{-1/2} \otimes \mathcal{L} \oplus \mathcal{K}^{1/2} \otimes \mathcal{L}$ we can decompose any spinor ψ as $\psi = \alpha \oplus \beta$ and in terms of this decomposition the Seiberg-Witten equations can be rephrased as

$$\left\{ \begin{array}{ll} \mathbf{i}\nabla_\zeta^A \alpha + \flat \bar{\nabla}^A \beta & + \lambda \alpha = 0 \\ \flat \nabla^A \alpha - \mathbf{i}\nabla_\zeta^A \beta & + \lambda \beta = 0 \\ \frac{1}{2}(|\alpha|^2 - |\beta|^2) & = \mathbf{i}F_A(\zeta_1, \zeta_2) \\ \mathbf{i}\alpha \bar{\beta} & = \bar{\varepsilon} \otimes F_A(\zeta_1 + \mathbf{i}\zeta_2, \zeta) = -F_A^{0,1} \end{array} \right. \quad (1.9)$$

The adiabatic Seiberg-Witten equations are obtained from the above by replacing λ with $\lambda/2$.

The Seiberg-Witten equations have a variational nature. Fix a smooth connection A_0 on \mathcal{L} and define

$$\mathbf{f} : L^{1,2}(\mathbb{S} \oplus \mathbf{i}T^*N) \rightarrow \mathbb{R}$$

by

$$\mathbf{f}(\psi, a) = \frac{1}{2} \int_N a \wedge (F_{A_0} + F_{A_0+a}) + \frac{1}{2} \int_N \langle \psi, \mathcal{D}_{A_0+a} \psi \rangle dv_h.$$

The differential of \mathbf{f} at a point $\mathbf{c} = (\phi, a)$ is

$$d\mathbf{c}\mathbf{f}(\dot{\phi}, \dot{a}) = \int_N \langle \dot{a}, \mathbf{c}^{-1}(\tau(\phi)) - *F_{A_0+a} \rangle dv_g + \int_N \Re \langle \dot{\phi}, \mathcal{D}_{A_0+a} \phi \rangle dv_g.$$

Hence, the solutions of (SW) are critical points of \mathbf{f} . Similarly, the solutions of $(SW)_\infty$ are critical points of

$$\mathcal{E}(\psi, a) = \frac{1}{2} \int_N a \wedge (F_{A_0} + F_{A_0+a}) + \frac{1}{2} \int_N \langle \psi, \mathbf{D}_{A_0+a} \psi \rangle dv_h$$

The solutions of $(SW)_{\infty, \delta}$ are critical points of a functional \mathcal{E}_δ , defined as \mathcal{E} but in terms of the metric h_δ .

Obviously, the adiabatic equations are zeroth order perturbations of the original ones. It is less obvious though, that the solutions of $(SW)_\delta$ converge as $\delta \rightarrow \infty$ to the solutions of $(SW)_\infty$ (see [10]). The remarkable feature of the adiabatic equations is that *they can be solved explicitly*, due mainly to the identity (1.6). In fact, for each δ we get a space of solutions of $(SW)_{\infty, \delta}$ but remarkably, these spaces turn out to be *independent of δ* !

To present the explicit description of the solutions of $(SW)_\infty$ let us first identify the space of $spin^c$ structures on N with $H^2(N, \mathbb{Z})$. Using the Thom-Gysin exact sequence for the S^1 -bundle $N \xrightarrow{\pi} \Sigma$ we deduce

$$H^2(N) \cong H^1(\Sigma, \mathbb{Z}) \oplus \pi^* H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_{|\ell|}.$$

Denote by $\mathcal{S}_\infty = \mathcal{S}_\infty(\mathcal{L})$ the space of solutions of SW_∞ modulo the action of the gauge group $\mathfrak{G} = \text{Aut}(L)$. In [10] we proved the following facts. We assume $\ell \neq 0$.
A. If $c_1(\mathcal{L}) \in H^2(N, \mathbb{Z})$ is not a torsion class then $\mathcal{S}_\infty = \emptyset$.

B. If $c_1(\mathcal{L})$ is a torsion class, $c_1 = \kappa \in \mathbb{Z}_{|\ell|}$, then \mathcal{E} is \mathfrak{G} -invariant (!) and the adiabatic moduli space \mathcal{S}_∞ consists of the following.

- A reducible part \mathcal{S}_∞^{red} which is homeomorphic with a torus T^{2g} . A pair (ψ, A) lies in \mathcal{S}_∞^{red} if and only if $\psi = 0$ and A is a flat connection on \mathcal{L} with holonomy along the fibers $\exp(2\pi \mathbf{i} \kappa / \ell)$.
- An irreducible part \mathcal{S}_∞^{irr} . Set $I_\kappa = \{n \in \mathbb{Z} ; 0 < |n| \leq g-1, n \equiv \kappa \bmod \ell\}$. $(\alpha \oplus \beta, A)$ lies in \mathcal{S}_∞^{irr} if and only if the following happen.

(i) There exists a complex line bundle $L \rightarrow \Sigma$ and a connection A_Σ on L such that $\deg L \in I_\kappa$, $\pi^*L = \mathcal{L}$ and $\pi^*A_\Sigma = A$.

(ii) If $\deg L < 0$ then $\alpha = 0$ and β is the pullback of a holomorphic section of $K^{1/2} \otimes L$. Moreover,

$$\frac{1}{4\pi} \|\beta\|^2 = -\deg L. \quad (1.10)$$

(iii) If $\deg L > 0$ then $\beta = 0$ and α is the pullback of an antiholomorphic section of $K^{-1/2} \otimes L$. Moreover,

$$\frac{1}{4\pi} \|\alpha\|^2 = \deg L. \quad (1.11)$$

The above facts show that if $(\psi, A) \in \mathcal{S}_\infty^{irr}$ then A induces on \mathcal{L} a structure of S^1 -equivariant line bundle and A is an S^1 -equivariant connection. As such, it has an S^1 -equivariant first Chern class $\hat{c}_1(\mathcal{L}, A) \in H_{S^1}^2(N, \mathbb{R})$ defined using the equivariant Chern-Weil theory described e.g. in [3]. A classical theorem of H. Cartan (see [11]) identifies $H_{S^1}^2(N) \cong H^2(\Sigma) \cong \mathbb{R}$. If $(\mathcal{L}, A) = \pi^*(L_\Sigma, A_\Sigma)$ (as in (i) above) then a simple computation shows that via the above identification we have

$$\hat{c}_1(\mathcal{L}, A) = \deg L_\Sigma$$

Hence, the number $\deg L_\Sigma$ is an *invariant* of the gauge equivalence class of the solution (ψ, A) of the adiabatic SW equations. We will denote this number by $\hat{d}(\psi, A)$. We get a continuous map

$$\hat{d} : \mathcal{S}_\infty^{irr} \rightarrow I_\kappa. \quad (1.12)$$

The fibers of this map are *connected* spaces, each homeomorphic to a symmetric product of a certain number of copies of Σ . More precisely, for $g \geq 2$ define

$$\nu : I_\kappa \rightarrow \mathbb{Z}_+$$

by

$$\nu(t) = (g-1) - |t|.$$

The space $\hat{d}^{-1}(t)$ is homeomorphic with a symmetric product of $\nu(t)$ copies of Σ . These spaces are spaces of *abelian vortices* on Σ . Note that $\mathcal{S}_\infty^{irr} = \emptyset$ for $g = 0, 1$.

As we have mentioned, for each δ we get different adiabatic equations $(SW)_{\infty, \delta}$ leading to the same moduli spaces. The energy functional \mathcal{E}_δ changes with δ and in [10] we showed the following.

Theorem 1.1 For δ sufficiently large, \mathcal{S}_∞^{irr} is the *nondegenerate critical set* of \mathcal{E}_δ (modulo \mathfrak{G}).

The goal of this paper is to understand the L^2 -gradient flow of \mathcal{E}_δ for $\delta \gg 1$.

§1.3 The 4-dimensional Seiberg-Witten equation on tubes Consider the tube $X = \mathbb{R} \times N$ and equip it with the product orientation. Next, fix a $spin^c$ structure σ on X and denote by $\hat{\mathbb{S}}_\sigma = \hat{\mathbb{S}}_\sigma^+ \oplus \hat{\mathbb{S}}_\sigma^-$ the associated bundle of spinors. Any connection A on $\det \hat{\mathbb{S}}_\sigma^+$ determines a Dirac operator

$$\mathfrak{D}_A : C^\infty(\hat{\mathbb{S}}_\sigma^+) \rightarrow C^\infty(\hat{\mathbb{S}}_\sigma^-).$$

The unknowns of the 4-dimensional Seiberg-Witten equations are an even spinor $\psi \in \Gamma(\hat{\mathbb{S}}_\sigma^+)$ and a connection A on $\det \hat{\mathbb{S}}_\sigma^+$ such that

$$\begin{cases} \mathfrak{D}_A \psi &= 0 \\ \frac{1}{2} \hat{\mathbf{c}}(F_A^+) &= \tau(\psi) \end{cases}$$

where $F_A^+ = 2^{-1}(F_A + \hat{*}F_A)$, $\hat{*}$ is the Hodge $*$ operator on X defined by the metric $dt^2 + h$, $\hat{\mathbf{c}} : \Omega^*(X) \otimes \mathbb{C} \rightarrow \text{End}(\hat{\mathbb{S}}_\sigma)$ denotes the Clifford multiplication map and

$$\tau(\psi) = \bar{\psi} \otimes \psi - \frac{1}{2} |\psi|^2 \in \text{End}(\hat{\mathbb{S}}_\sigma^+) \subset \text{End}(\hat{\mathbb{S}}_\sigma).$$

The factor $1/2$ (which can be eventually absorbed by rescaling ψ) is not traditionally present in the equations but will help match the conventions of this section with those in §1.2 where we work with connections on $\det^{1/2} \mathbb{S}_\sigma$.

Using the special product structure of the tube X we can further transform these equations. First of all, note that there is a bijection between the $spin^c$ structures on X and those on N . Fix one such structure on N and denote by \mathbb{S}_σ the associated spinor bundle and by \mathbf{c} the corresponding Clifford multiplication map. Along the slices $\{t\} \times N$ we have

$$\hat{\mathbb{S}}_\sigma^\pm \cong \mathbb{S}_\sigma$$

and

$$\mathbf{c}(\omega) = \hat{\mathbf{c}}(dt)\hat{\mathbf{c}}(\omega) \quad \forall \omega \in \Omega^1(N). \quad (1.13)$$

We have seen that the line bundle $L_\sigma = \det \mathbb{S}_\sigma \rightarrow N$ has a natural square root \mathcal{L}_σ . Fix a connection A_0 on the line bundle \mathcal{L}_σ . Modulo a gauge transformation, any connection on $\mathbb{R} \times \mathcal{L}_\sigma$ can be written as $A = A_0 + a(t)$, where for each t , $a(t)$ is a purely imaginary 1-form on N . Thus A can be regarded as a path of connections $A(t) = A_0 + a(t)$ on \mathcal{L}_σ . Denote by \hat{F}_A the curvature of A on X and, for each t , denote by F_a the curvature of $A_0 + a(t)$ on $\{t\} \times N$. Then

$$\hat{F}_A = F_{a(t)} + dt \wedge \dot{a}$$

A simple computation shows that

$$\hat{F}_A^+ = \frac{1}{2} \{ (F_a + * \dot{a}(t)) + dt \wedge (\dot{a}(t) + * F_a) \}$$

Using the equality (1.13) we deduce that

$$\begin{aligned}\hat{\mathbf{c}}(\hat{F}_A^+) &= \mathbf{c}(F_a + *\dot{a}(t)) \\ \mathfrak{D}_A &= \hat{\mathbf{c}}(dt) \left\{ \frac{\partial}{\partial t} - \mathcal{D}_A \right\}.\end{aligned}$$

A spinor $\psi \in \Gamma(\hat{\mathbb{S}}_\sigma^+)$ can be regarded as a path of spinors $\psi(t) \in \Gamma(\mathbb{S}_\sigma|_{\{t\} \times N})$. Putting together all of the above we deduce that the Seiberg-Witten equation on a tube can be rewritten as equations for a path $A = A_0 + a(t)$ of connections on \mathcal{L}_σ and a path of spinors $\psi(t)$. More exactly, they are

$$\begin{cases} \dot{\psi} &= \mathcal{D}_A \psi \\ \mathbf{c}(\dot{a}) &= \tau(\psi) - \mathbf{c}(*F_a) \end{cases} \quad (1.14)$$

These equations describe the L^2 positive gradient flow of \mathfrak{f} . We can now define *the adiabatic gradient flow* in an obvious fashion.

$$\begin{cases} \dot{\psi} &= \mathbf{D}_A \psi \\ \mathbf{c}(\dot{a}) &= \tau(\psi) - \mathbf{c}(*F_a) \end{cases} \quad (1.15)$$

We will be interested exclusively in *finite energy* solutions (or *tunnelings*) of (1.15). These are solutions $(\psi(t), A(t))$ satisfying $\psi(t) \in L_{loc}^{1,2} \cap L_{loc}^\infty$, $A(t) \in L_{loc}^{1,2}$ and

$$\int_{-\infty}^{\infty} \|\dot{\psi}(t)\|^2 + \|\dot{a}(t)\|^2 dt < \infty$$

Intuitively, the tunnelings have limits as $t \rightarrow \pm\infty$ which should be solutions of the 3-dimensional (adiabatic) Seiberg-Witten equations.

The Weitzenböck formula for \mathfrak{D}_A has a special form on the tube. It splits in two parts. The first part is the Weitzenböck formula for \mathcal{D}_A

$$\mathcal{D}_A^2 = \nabla_A^* \nabla_A + \mathbf{c}(F_A) + \frac{s}{4} \quad (1.16)$$

(where s denotes the scalar curvature of N) and the second part is

$$[\nabla_t, \mathcal{D}_{A(t)}] := \nabla_t \mathcal{D}_{A(t)} - \mathcal{D}_{A(t)} \nabla_t = \mathbf{c}(\dot{a}) \quad (1.17)$$

where ∇_t denotes the t -derivative with respect to the connection A_0 on $\mathbb{R} \times \mathcal{L}_\sigma$.

2 Tunnelings

In this section we study the tunnelings of (1.15). More precisely, we will prove the following.

Theorem 2.1 If $\delta \gg 1$ then the equation $(1.15)_\delta$ has no tunnelings.

The proof will be carried out in three steps.

Step 1 Produce L_{loc}^4 estimates *independent of δ* for the spinor component of (1.15). This will follow from the Weitzenböck formula coupled with some *universal* energy estimates.

Step 2 Produce *effective* L_{loc}^∞ estimates of the spinor component in terms of the above L_{loc}^4 estimates and the best Sobolev constant of the embedding $L^{1,2} \hookrightarrow L^4$ on a cylinder $[T, T+4] \times N$. This is achieved via the Möser's iteration technique; see [2] or [5].

Step 3 Conclude the proof using the estimates in [2] of the best Sobolev constants in terms of the background geometry.

We will now supply the details.

§2.1 L_{loc}^4 estimates We will use the metric h_δ on N and in the equations (1.15) all the intervening quantities should expressed in terms of this metric. In the sequel, we will carefully keep track of this dependence which will be signaled by δ subscripts.

For each $T \in \mathbb{R}$ and $L > 0$ and for any tunneling $(\psi(t), a(t))$ of (1.15) set

$$E_\delta(T, L) = E_\delta(\psi, A, T, L) = \frac{1}{2} \int_{T-L}^{T+L} dt \int_N |\dot{a}(t)|_\delta^2 + |\dot{\psi}(t)|^2 dv_\delta.$$

A simple integration by parts, using (1.15) yields the following identity.

$$E_\delta(T, L) = \mathcal{E}_\delta(\psi(T+L), A_0 + a(T+L)) - \mathcal{E}_\delta(\psi(T-L), a(T-L)). \quad (2.1)$$

Recall the following result of [8].

Lemma 2.2 If $(\psi(t), a(t))$ is a finite energy solution of (1.15) then there exist subsequences $t_n^\pm \rightarrow \pm\infty$ and solutions $(\psi_\pm, A_\pm) \in \mathcal{S}_\infty(\mathcal{L}_\sigma)$ such that (modulo $\text{Aut}(\mathcal{L}_\sigma)$) we have

$$\lim_{t_n^\pm \rightarrow \pm\infty} (\psi(t_n^\pm), A(t_n^\pm)) = (\psi_\pm, A_\pm)$$

in the strong $L^{1,2}$ -topology.

The above lemma implies that if $c_1(\mathcal{L}_\sigma)$ is not a torsion class then there exist no finite energy solutions of (1.15). In the sequel, we will assume the first Chern class is torsion. In this case the functional \mathcal{E} is \mathfrak{G} -invariant the following quantity is well defined and finite.

$$G_\delta = \max\{\mathcal{E}_\delta(\psi_+, A_+) - \mathcal{E}_\delta(\psi_-, A_-) ; (\psi_\pm, A_\pm) \in \mathcal{S}_\infty(\mathcal{L}_\sigma)\}.$$

G_δ is the largest energy gap between the various components of the critical set of \mathcal{E}_δ . Using again the gauge invariance of \mathcal{E} , Lemma 2.2 and (2.1) we deduce

$$\forall \text{ tunneling, } \forall T \in \mathbb{R}, \forall L > 0 \quad E_\delta(\psi, A, T, L) \leq G_\delta. \quad (2.2)$$

Recall that the definition of \mathcal{E}_δ depended on a reference connection A_0 . We now choose A_0 to be a *flat* connection on \mathcal{L}_σ . Any other connection has the form $A = A_0 + a$, $a \in \Omega^1(N)$. We can now rewrite

$$\mathcal{E}_\delta = \frac{1}{2} \left\{ \int_N a \wedge da + \int_N \langle \psi, \mathbf{D}_{A_0+a, \delta} \psi \rangle dv_\delta \right\}. \quad (2.3)$$

We have the following remarkable result.

Lemma 2.3 There exists an universal constant G such that

$$G_\delta \leq G \quad \forall \delta \geq 1.$$

Proof The estimate follows from a direct computation of the energy on the components of \mathcal{S}_∞ . Note that on \mathcal{S}_∞^{red} we have $\mathcal{E}_\delta \equiv 0$.

Along the components of \mathcal{S}_∞^{irr} , the second term in the definition of \mathcal{E}_δ vanishes (since $\mathbf{D}_A \psi = 0$) while the first term is independent of δ . More explicitly, the energy along the component of \mathcal{S}_∞^{irr} given by the fiber $\hat{d}^{-1}(n)$ (see (1.12)) is $\frac{2\pi^2 n^2}{\ell}$. To see this note that in this case

$$\mathcal{E}_\delta(\psi, A_0 + a) = \frac{1}{2} \int_N a \wedge da.$$

Since $F_{A_0+a}^{0,1} = 0$ we can pick a of the form $\mathbf{i}(t\eta + \pi^* a')$ where $a' \in \Omega^1(\Sigma)$ and $t \in \mathbb{R}$. We get

$$\mathcal{E}_\delta(\psi, A_0 + a) = -\frac{t^2}{2} \int_N \eta \wedge d\eta - \frac{t}{2} \int_N \eta \wedge da'$$

Integrating along fibers and using $\int_{N/\Sigma} \eta = 2\pi$ we get

$$\mathcal{E}_\delta(\psi, A_0 + a) = \pi t^2 \int_\Sigma 2\ell dv_\Sigma - \pi t \int_\Sigma da' = 2\pi^2 t^2 \ell.$$

On the other hand,

$$n = \hat{d}(\psi, A_0 + a) = \hat{c}_1(A_0 + a) = \frac{\mathbf{i}}{2\pi} \int_\Sigma \mathbf{i}(td\eta + da') = -\frac{-t}{2\pi} \int_\Sigma (-2\ell) dv_\Sigma = \ell t.$$

Hence

$$\mathcal{E}_\delta(\psi, A_0 + a) = \frac{2\pi^2 n^2}{\ell}.$$

Lemma 2.3 is proved. \square

To prove the promised L_{loc}^4 estimates we need the following energy identity.

Lemma 2.4 If $(\psi(t), a(t))$ is a tunneling of (1.15) then for every $T \in \mathbb{R}$ and any $L > 0$ we have

$$\begin{aligned} & \int_{T-L}^{T+L} dt \int_N |\dot{\psi}|^2 + |\dot{a}|^2 dv_\delta \\ &= \int_{T-L}^{T+L} dt \int_N \left(|\nabla^{A(t)} \psi|_\delta^2 + \frac{1}{8} |\psi|^4 + \frac{1}{4} (s_\delta - \lambda_\delta^2) |\psi|^2 - \lambda_\delta \langle \dot{\psi}, \psi \rangle + |F_A|_\delta^2 \right) dv_\delta. \end{aligned} \quad (2.4)$$

Above $A(t)$ denotes the connection $A_0 + a(t)$ on the slice $\{t\} \times N$ and s_δ denotes the scalar curvature of the metric h_δ .

To keep the flow of arguments uninterrupted we present the proof of this lemma in an appendix.

We can now produce the promised L^4_{loc} estimates. Set $C(T, L) = [T - L, T + L] \times N$. In [10] we showed that $\sup_{x \in N} |s_\delta(x)|$ is bounded from above as $\delta \rightarrow \infty$. Denote by $|s|$ such an upper bound. The energy identity implies that there exists a constant $C > 0$ independent of $\delta \gg 1$ such that

$$\begin{aligned} \int_{C(T,L)} |\psi|^4 dt dv_\delta &\leq C \left(E(T, L) + \int_{C(T,L)} |\psi|^2 dt dv_\delta \right) \\ &\leq C \left\{ E(T, L) + \left(\frac{L}{\delta} \int_{C(T,L)} |\psi|^4 dt dv_\delta \right)^{1/2} \right\}. \end{aligned}$$

The last inequality implies

$$\int_{C(T,L)} |\psi|^4 dt dv_\delta \leq C(E(T, L) + L\delta^{-1}).$$

Using the inequality (2.3) we deduce that there exists a constant C independent of δ such that

$$\int_{C(T,L)} |\psi|^4 dt dv_\delta \leq C(1 + L\delta^{-1}). \quad (2.5)$$

§2.2 L^∞ estimates In the compact case, effective L^∞ estimates can be obtained using a simple maximum principle trick as in [7]. In our noncompact situation the M\"oser's iteration technique produces results which are dramatically sharper in the adiabatic limit. The starting point is the following Kato type inequality, very similar to the one used in [7].

Lemma 2.5 If $(\psi(t), A(t))$ is a solutions of $(1.15)_\delta$ then

$$\Delta_\delta |\psi|^2 + 2\lambda_\delta \frac{\partial |\psi|^2}{\partial t} \leq (\lambda_\delta + z_\delta) |\psi|^2 \quad (2.6)$$

where Δ_δ denotes the scalar Laplacian of the metric $dt^2 + h_\delta$ and $z_\delta = \sup_{x \in N} |s_\delta(x)|$.

Proof For simplicity, we prove the above inequality only for $\delta = 1$. We first rewrite the first equation in (1.15) using the operator \mathfrak{D}_A . We get

$$\mathfrak{D}_A \psi = \frac{\lambda}{2} \hat{\mathbf{c}}(dt) \psi.$$

We now apply \mathfrak{D}_A^* to both sides of the above equality and we obtain

$$\mathfrak{D}_A^* \mathfrak{D}_A \psi = \frac{\lambda}{2} \mathfrak{D}_A^* \hat{\mathbf{c}}(dt).$$

The anticommutation equality Prop. 3.45 in [3],

$$\mathfrak{D}_A^* \hat{\mathbf{c}}(dt) + \hat{\mathbf{c}}(dt) \mathfrak{D}_A^* = -2\nabla_t$$

yields

$$\mathfrak{D}_A^* \mathfrak{D}_A \psi = \frac{\lambda^2}{4} \psi - \lambda \dot{\psi}.$$

Using the Weitzenböck formula in the left-hand side of the above inequality we deduce

$$(\nabla_A^* \nabla_A + \hat{\mathbf{c}}(F_A^+) + \frac{s}{4} \psi = \frac{\lambda^2}{4} \psi - \lambda \dot{\psi}.$$

The inequality follows as in [7], by taking the (pointwise) inner product with ψ , then using the equality $\hat{\mathbf{c}}(F_A^+) = \tau(\psi)$ to eliminate a positive multiple of $|\psi|^4$ and concluding with the Kato inequality (see [2])

$$\Delta |\psi|^2 \leq 2 \langle \nabla_A^* \nabla_A \psi, \psi \rangle. \quad \square$$

Denote by S_δ the best Sobolev constant defined as the largest constant such that

$$S_\delta \|u\|_{4,\delta}^2 \leq (\|du\|_{2,\delta}^2 + \|u\|_{2,\delta}^2), \quad \forall u \in C_0^\infty(C(T, 2)). \quad (2.7)$$

Above, $\|\cdot\|_{p,\delta}$ denotes the L^p norm on $C(T, 2)$ defined in terms of the metric $dt^2 + h_\delta$.

We can now state the main result of this subsection.

Lemma 2.6 There exists a constant $C > 0$ independent of δ such that if (ψ, A) is a tunneling of $(1.15)_\delta$ then

$$\sup_{(t,x) \in C(T,1)} |\psi(t, x)|^2 \leq \frac{C}{S_\delta} \|\psi\|_{4,\delta}^2.$$

Proof Set $u(t, x) = |\psi(t, x)|^2$. Then $u \in L_{loc}^{1,2}(\mathbb{R} \times N)$ and satisfies the differential inequality

$$\Delta_\delta u + 2\lambda_\delta \dot{u} \leq Au \text{ in } C(T, 2) \quad (2.8)$$

where A is a positive constant independent of δ . This is precisely the type of inequality for which the M\"oser iteration technique-as described e.g. in [5]- is applicable. This leads to an inequality of the type

$$\sup_{(t,x) \in C(T,1)} u(t, x) \leq B \|u\|_{2,\delta}.$$

To understand the manner in which the constant B depends upon δ we will go carefully through the proof. To make the presentation more accessible we will omit the subscript δ in the notations of the various norms and in the notation of the volume form. We will also assume $\lambda = 0$ in (2.8) since this term can be later absorbed anyway via a simple interpolation trick.

For each $h \in (0, 1)$ consider a nonnegative cutoff function

$$f = f_h(t) \in C_0^\infty(T - 2, T + 2)$$

identically 1 on $(T - 2 + h, T + 2 - h)$ and such that $|\sup f'(t)| \leq 4h^{-1}$. We set for brevity $M = C(T, 2) = [T - 2, T + 2] \times N$ and $M_h = C(T, 2 - h)$.

Multiply the equality (2.8) by $v = f^2 u^{2p-1}$ where p will be specified latter. Using

$$dv = 2f u^{2p-1} df + (2p - 1) f^2 u^{2p-2} du.$$

we obtain after an integration by parts

$$\begin{aligned} \int_M (2p - 1) f^2 u^{2p-2} |du|^2 &\leq A \int_M f^2 u^{2p} + \int_M 2f u^{2p-1} |du| |df| \\ &= A \int_M (2p - 1) f^2 u^{2p-2} |du|^2 + \int_M 2f u^{p-1} |du| \cdot u^p |df| \\ &\leq A \int_M (2p - 1) f^2 u^{2p-2} |du|^2 + \varepsilon \int_M f^2 u^{2p-2} |du|^2 + \varepsilon^{-1} \int_M u^2 |df|^2. \end{aligned}$$

If we take $\varepsilon = 1/2$ and absorb the corresponding term in the left-hand-side we get

$$(2p - 2) \int_M f^2 u^{2p-2} |du|^2 \leq A_1 \int_M u^{2p} \quad (2.9)$$

where A_1 is a constant *independent* of δ . Now set $w = f u^p$ so that $\text{supp } w \subset \text{supp } f$ and on M

$$|dw|^2 \leq 2(u^{2p} |df|^2 + p^2 f^2 u^{2p-2} |df|^2).$$

Hence, using (2.9) we deduce

$$\int_M |dw|^2 + |w|^2 \leq A_2 p h^{-2} \int_M u^{2p}$$

where A_2 is independent of δ . The Sobolev inequality (2.7) now implies

$$S_\delta \|w\|_{4,M}^2 \leq A_2 p h^{-2} \int_M u^{2p}.$$

Hence

$$\left(\int_{M_h} u^{4p} \right)^{1/2} \leq \frac{A_2 p}{S_\delta h^2} \int_M u^{2p}$$

so that

$$\left(\int_{M_h} u^{4p} \right)^{1/4p} \leq \left(\frac{A_2 p}{S_\delta h^2} \right)^{1/2p} \left(\int_M u^{2p} \right)^{1/2p}.$$

More generally, for each $0 \leq k < h < 1$ we have a similar inequality

$$\left(\int_{M_h} u^{4p} \right)^{1/4p} \leq \left(\frac{z p}{S_\delta (h-k)^2} \right)^{1/2p} \left(\int_{M_k} u^{2p} \right)^{1/2p} \quad (2.10)$$

where z is a positive constant independent of δ , p , h , k . The above inequality is the basis of the M\"oser iterative method. Set $p_0 = 2$, $p_{n+1} = 2p_n = 2^{n+1}$, $h_0 = 0$, $h_{n+1} = h_n + 2^{-n-1}$. We deduce

$$\|u\|_{p_{n+1}, M_{h_{n+1}}} \leq \left(\frac{z 2^n}{S_\delta (h_{n+1} - h_n)^2} \right)^{1/2^n} \|u\|_{p_n, M_{h_n}} = \left(\frac{z 2^{3n+2}}{S_\delta} \right)^{1/2^{n+1}} \|u\|_{p_n, M_{h_n}}$$

Iterating and then letting $n \rightarrow \infty$ we deduce

$$\sup_{M_1} u(x) \leq \frac{z}{S_\delta} \left(\prod_{n=0}^{\infty} 2^{(3n+2)/2^{n+1}} \right) \cdot \|u\|_{2,M}.$$

Lemma 2.6 is proved. \square

§2.3 Proof of the Theorem 2.1 We will argue by contradiction. Assume that for all $\delta \gg 1$ there exists a tunneling $(\psi(t), A(t)) = (\psi_\delta(t), A_\delta(t))$. Since the Ricci curvature of M is bounded as $\delta \rightarrow \infty$ and $\text{diam}(M) = 4$ we deduce from the estimates in [2] (formula (2) p. 389 coupled with Theorem 2 p. 386) that

$$S_\delta = O(\text{vol}(N, h_\delta)^{-1/2}) = O(\delta^{1/2}) \text{ as } \delta \rightarrow \infty.$$

Lemma 2.6 coupled with (2.5) yields

$$\|\psi\|_{\infty, C(T,2)} = O(\delta^{-1/2}) \quad \forall T.$$

Lemma 2.2 now implies

$$\|\psi_\pm\|_2 = O(\delta^{-1/2}) \quad (2.11)$$

where $\|\cdot\|$ denotes the L^2 norm on N with respect to the metric h_1 . At least one of the limits ψ_\pm is nontrivial since the reducible part of the critical set is path connected.

On the other hand, by (1.10) or (1.11) we deduce that if ψ_{\pm} is nontrivial, then its L^2 norm belongs to a finite set of *universal constants*. This contradicts (2.11) and concludes the proof of Theorem 2.1. \square

Remark 2.7 (a) The result we have proved suggests that \mathcal{E}_{δ} is a perfect Morse function. To determine the *SW*-Floer (co)homology we must determine the spectral flows of the Hessian along paths connecting different components of the critical sets which determine the grading of this homology. We believe it is isomorphic (as a *graded* group) with the homology of the irreducible part of the adiabatic moduli space. We will deal with this issue elsewhere. We refer also to [9] for an application along these lines.

(b) There exists a different approach to the study of tunnelings similar in spirit with the study of Yang Mills tunnelings in [6]. More precisely, using (1.6) and the exponential decay (which is valid if the fibers are sufficiently short as a consequence of Theorem 1.1) one can identify the tunnelings between two *irreducible* components of the critical set with some abelian vortices over the complex ruled surface X compactifying $\mathbb{R} \times N$. One then can use algebraic geometric techniques to study these vortices. We refer to [9] where this approach is dicussed in great detail.

A Proof of Lemma 2.4

We prove the identity for $\delta = 1$. Note that $\mathbf{D}_A^2 = \mathcal{D}_A^2 - \lambda \mathcal{D}_A + \lambda^2/4$. Hence

$$\int_N |\dot{\psi}|^2 = \int_N |\mathbf{D}_A \psi|^2 = \int_N \langle \mathcal{D}_A^2 \psi, \psi \rangle - \lambda \int_N \Re \langle \mathcal{D}_A \psi, \psi \rangle + \frac{\lambda^2}{4} \int_M |\psi|^2.$$

Using the Weitzenböck formula for \mathcal{D}_A^2 we deduce

$$\int_N |\dot{\psi}|^2 = \int_N \left(|\nabla^A \psi|^2 + \frac{s(x)}{4} |\psi|^2 + \Re \langle \mathbf{c}(F_A) \psi, \psi \rangle \right) - \lambda \int_N \Re \langle \mathcal{D}_A \psi, \psi \rangle + \frac{\lambda^2}{4} \int_M |\psi|^2.$$

On the other hand,

$$\begin{aligned} 2 \int_N |\dot{A}|^2 &= \int_N |\mathbf{c}(\dot{A})|^2 = \int_N |\tau(\psi)|^2 + |\mathbf{c}(F_A)|^2 - 2 \int_N \Re \langle \mathbf{c}(F_A), \tau(\psi) \rangle \\ &= 2 \int_N |F_A|^2 + \frac{1}{4} \int_N |\psi|^4 - 2 \int_N \Re \langle \mathbf{c}(F_A), \tau(\psi) \rangle. \end{aligned}$$

A simple computation yields

$$\Re \langle \mathbf{c}(F_A), \tau(\psi) \rangle = \Re \langle \mathbf{c}(F_A) \psi, \psi \rangle.$$

We conclude that

$$\int_N |\dot{\psi}|^2 + |\dot{A}|^2$$

$$\begin{aligned}
&= \int_N \left(|\nabla^A \psi|^2 + \frac{s(x)}{4} |\psi|^2 + |F_A|^2 + \frac{1}{8} |\psi|^4 \right) - \lambda \int_N \Re \langle \mathcal{D}_A \psi, \psi \rangle + \frac{\lambda^2}{4} \int_M |\psi|^2 \\
&= \int_N \left(|\nabla^A \psi|^2 + \frac{s(x)}{4} |\psi|^2 + |F_A|^2 + \frac{1}{8} |\psi|^4 \right) - \lambda \int_N \Re \langle \dot{\psi} + \frac{\lambda}{2} \psi, \psi \rangle + \frac{\lambda^2}{4} \int_N |\psi|^2 \\
&= \int_N \left(|\nabla^A \psi|^2 + \frac{s(x)}{4} |\psi|^2 + |F_A|^2 + \frac{1}{8} |\psi|^4 \right) - \frac{\lambda^2}{4} \int_N |\psi|^2 - \lambda \int_N \Re \langle \dot{\psi}, \psi \rangle.
\end{aligned}$$

This concludes the proof of the energy identity. \square

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